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DETERMINATION OF NONSTATIONARY SURFACE DISTORTIONS OF AN AXISYMMETRICALLY HEATED HALF-SPACE BY THE GREEN'S FUNCTION METHOD

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The nonstationary normal displacements and the surface temperature of an elastic half-space heated in a circular region by a constant-power heat flux are evaluated.

Determination of the thermal distortion of surface profiles enters the analysis of contact problems of thermoelasticity, in particular, when heat is generated in the region where the bodies come into contact. Stresses and displacements in an elastic half-space with anarbitrary stationary temperature distribution are evaluated in some works, e.g., [1]. However, the change in the contact pressure and the contact area upon heat generation due to friction is a highly nonstationary process [2]. It is known that the solution of the equations of nonstationary heat distribution for a semi-infinite body may be represented as a double integral over space and time variables. Discretization of this representation by numerical methods, in particular by the method of finite elements, requires the construction of the corresponding Green's functions. The present work is devoted to investigating precisely this problem under the assumption of a constant distribution of heat fluxes in some small circular region on the half-space surface.

Instantaneous Sources. We will determine normal displacements of the surface of the elastic half-space $z > 0$ heated by a constant-power heat flux q instantaneously applied to it at the point O at the time $t = 0$. Hereinafter we assume that the initial temperature of the semi-infinite body is zero and the surface $z = 0$ outside the heating region is heat-insulated. At $t > 0$ at the distance $r = \sqrt{x^2 + y^2}$ from the point O we have [3]

$$T(r,\ t) = \frac{q}{4\rho c\,(\pi k t)^{3/2}}\,e^{-R^2}, \quad R^2 = \frac{r^2}{4kt}. \tag{1}$$

Temperature field (1) gives rise to a stress-strain state whose components in the spherical system of coordinates r, θ, φ are equal to [4]:

$$\sigma_{rr}(r,\ t) = -\frac{2\alpha E}{(1-\nu)\,r^3}\int_0^r T(s,\ t)\,s^2 ds =$$

$$= \frac{\alpha E q}{8\,(1-\nu)\,\rho c\,(\pi k t)^{3/2}}\left[\frac{2Re^{-R^2} - \sqrt{\pi}\,\mathrm{erf}\,(R)}{R^3}\right], \tag{2}$$

The shear stresses $\sigma_{\theta\varphi}$, $\sigma_{r\varphi}$ and the displacements u_0 are equal to zero as a consequence of the loading symmetry.

Solution (2) corresponds to the case where the surface $z = 0$ remains two-dimensional due to the action of the stresses $\sigma_{\varphi\varphi}$ applied to it. We obtain a solution for the half-space $z > 0$ with a stress-free boundary by applying to the latter opposite, in sign, forces equal to $\sigma_{\varphi\varphi}$ from (2). Using the Boussinesq solution for a concentrated force acting at a boundary point of the half-space [4], we determine the normal displacements of the surface $z = 0$:

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$$\sigma_{\theta\theta}(r,t)=\sigma_{\varphi\varphi}(r,t)=\frac{\alpha E}{(1-\nu)}\left[\frac{1}{r^3}\int_0^r T(s,t)\,s^2ds-T(r,t)\right]=$$

$$=-\frac{\alpha Eq}{16(1-\nu)\rho c(\pi kt)^{3/2}}\left[\frac{2R\mathrm{e}^{-R^2}-\sqrt{\pi}\,\mathrm{erf}(R)}{R^3}+4\mathrm{e}^{-R^2}\right],$$

$$u_r(r,t)=\frac{\alpha(1+\nu)}{(1-\nu)r^2}\int_0^r T(s,t)\,s^2ds=$$

$$=-\frac{\alpha(1+\nu)q}{16(1-\nu)\rho c(\pi kt)^{3/2}}\left[\frac{2R\mathrm{e}^{-R^2}-\sqrt{\pi}\,\mathrm{erf}(R)}{R^3}\right].$$

$$u_z(r,t)=-\frac{(1-\nu^2)}{\pi E}\int_0^\infty\int_0^{2\pi}\frac{\sigma_{\varphi\varphi}(s,t)\,sd\theta ds}{\sqrt{r^2-2rs\cos\theta+s^2}}=$$

$$=\frac{q\delta}{8\pi^{5/2}t}\int_0^\infty\int_0^{2\pi}\frac{[2\mathrm{e}^{-S^2}/S+4\mathrm{e}^{-S^2}/S^2-\sqrt{\pi}\,\mathrm{erf}(S)/S^2]\,d\theta dS}{\sqrt{R^2-2RS\cos\theta+S^2}}=$$

$$=q\frac{\delta}{4\pi t}\Phi\left(\frac{3}{2},2;-R^2\right),\quad S^2=\frac{s^2}{4kt},\quad \delta=\frac{\alpha(1+\nu)}{k\rho c}. \tag{3}$$

To calculate the degenerate hypergeometric function $\Phi(3/2, 2; -R^2)$ for $R \le 1$, we use the power series [5]

$$\Phi\left(\frac{3}{2},2;-R^2\right)=2\sum_{i=0}^{\infty}\frac{(2i+1)!!\,(-R^2)^i}{(2i+2)!!\,i!}, \tag{4}$$

and for $R > 1$ the asymptotics [6]

$$\Phi\left(\frac{3}{2},2;-R^2\right)\cong\frac{1}{\sqrt{\pi}\,R}\sum_{i=0}^{N}\frac{(2i+1)!!\,(2i-1)!!}{4^i\,i!\,(R^2)^{i+1}}, \tag{5}$$

herein $(-1)!! \equiv 1$. Moreover, in [7] it is shown that

$$\Phi\left(\frac{3}{2},2;-R^2\right)=\mathrm{e}^{-X}[I_0(X)-I_1(X)],\quad X=\frac{R^2}{2}.$$

Let instantaneous sources with the power $qsd\theta$ be situated on a circle of radius s in the plane $z = 0$ and act at $t = 0$. Then at $t > 0$ at a point of the surface of the half-space at the distance r from the center of the circle, on the basis of (3) we have

$$u_z(r,t)=q\delta\sqrt{k/t}\,S\Phi_1(R,S),$$

$$\Phi_1(R,S)\equiv\frac{1}{2\pi}\int_0^{2\pi}\Phi\left(\frac{3}{2},2;-R^2+2RS\cos\theta-S^2\right)d\theta. \tag{6}$$

With account for (4), for the function $\Phi_1(R, S)$ we have the expansion

$$\Phi_1(R, S) = 2\sum_{i=0}^{\infty} \frac{(2i+1)!!\,(-S^2)^i}{(2i+2)!!\,i!} \sum_{j=0}^{i} (C_j^i)^2 \left(\frac{R}{S}\right)^{2j}, \tag{7}$$

$$C_j^i = \frac{(j)!\,(i-j)!}{i!}.$$

It is noteworthy that $\Phi_1(R, S) = \Phi_1(S, R)$ and $\Phi_1(0, S) = \Phi(3/2, 2; -S^2)$. The corresponding values of the surface temperature are presented in [3].

We assume that on a disk of radius a in the plane $z = 0$ the heat $\pi a^2 q$ is instantaneously generated at the moment $t = 0$. The thermal distortion of the initially plane surface $z = 0$ of the half-space with account for (6) is equal to

$$u_z(r, t) = q\delta k A^2 \Phi_2(R, A), \tag{8}$$

$$\Phi_2(R, A) \equiv \frac{2}{A^2}\int_0^A S\Phi_1(R, S)\,dS, \quad A^2 = \frac{a^2}{4kt}.$$

Substituting into (8) expansion (7) of the function $\Phi_1(R, S)$ in a power series, we arrive at

$$\Phi_2(R, A) = 2\sum_{i=0}^{\infty} \frac{(2i+1)!!\,(-A^2)^i}{(2i+2)!!\,i!} \sum_{j=0}^{i} (C_j^i)^2 \frac{(R/A)^{2j}}{(i-j+1)}, \quad R \le A,$$

$$\Phi_2(R, A) = 2\sum_{i=0}^{\infty} \frac{(2i+1)!!\,(-R^2)^i}{(2i+2)!!\,i!} \sum_{j=0}^{i} (C_j^i)^2 \frac{(R/A)^{2j}}{(j+1)}, \quad R > A.$$

We note that $\Phi_2(R, 0) = \Phi(3/2, 2; -R^2)$, while at the center of the heated disk we have $\Phi(0, A) = \Phi(3/2, 2; -A^2)$. The temperature of the boundary points of the surface of the half-space is determined by using formulas from [3].

Continuous Sources. We determine the displacements and temperature of the surface $z = 0$ for a point source continuously acting on it by integrating corresponding equations (3) and (1) for an instantaneous source over time. We have

$$u_z(r, t) = -q\frac{\delta}{2\pi}\Phi_3(R), \quad \Phi_3(R) \equiv -\int_0^{\infty} \Phi\left(\frac{3}{2}, 2; -S^2\right)\frac{dS}{S}, \tag{9}$$

$$T(r, t) \equiv \frac{q}{2\pi k\rho cr}\operatorname{erfc}(R). \tag{10}$$

With account for expansion (4), for the function $\Phi_3(R)$ we may write

$$\Phi_3(R) = \ln(R/2) + (1+\gamma)/2 + \sum_{i=1}^{\infty} \frac{(2i-1)!!\,(-R^2)^i}{(2i+2)!!\,i!}. \tag{11}$$

At large (> 1) values of R, from (9) and (5) we have

$$\Phi_3(R) \cong -\frac{1}{\sqrt{\pi}\,R}\sum_{i=0}^{N} \frac{(2i+1)!!\,(2i-1)!!}{(2i+3)\,4^i\,i!\,(R^2)^{i+1}}. \tag{12}$$

If the heat supply to the half-space is accomplished over a thin circular ring of radius s, then the distortion of the surface $z = 0$ of the semi-infinite body, according to (9), is
$u_z(r, t) = -q\delta 2\sqrt{kt}\, S\Phi_4(R, S)$,

$$\Phi_4(R, S) \equiv \frac{1}{2\pi}\int_0^{2\pi} \Phi_3(\sqrt{R^2 - 2RS\cos\theta + S^2})\, d\theta. \tag{13}$$

Integrating relation (10), we obtain the temperature distribution over the surface $z = 0$ of the half-space:

$$T(r, t) = \frac{qS}{k\rho c}\Phi_5(R, S),$$

$$\Phi_5(R, s) \equiv \frac{1}{2\pi}\int_0^{2\pi} \frac{\operatorname{erfc}[(R^2 - 2RS\cos\theta + S^2)^{1/2}]\, d\theta}{(R^2 - 2RS\cos\theta + S^2)^{1/2}}. \tag{14}$$

Taking into account (11), we obtain for the functions Φ_4, Φ_5 for $R \le S$

$$\Phi_4(R, S) = \ln(S/2) + (1+\gamma)/2 + \sum_{i=1}^{\infty} \frac{(2i-1)!!\,(-S^2)^i}{(2i+2)!!\, i!\, i} \sum_{j=0}^{i} (C_j^i)^2 \left(\frac{R}{S}\right)^{2j},$$

$$\Phi_5(R, S) = \frac{2}{\pi S} K\left(\frac{R}{S}\right) - \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-S^2)^i}{i!\,(2i+1)} \sum_{j=0}^{i} (C_j^i)^2 \left(\frac{R}{S}\right)^{2j} \tag{15}$$

and for $R > S$

$$\Phi_4(R, S) = \ln(R/2) + (1+\gamma)/2 + \sum_{i=1}^{\infty} \frac{(2i-1)!!\,(-R^2)^i}{(2i+2)!!\, i!\, i} \sum_{j=0}^{i} (C_j^i)^2 \left(\frac{S}{R}\right)^{2j},$$

$$\Phi_5(R, S) = \frac{2}{\pi S} K\left(\frac{S}{R}\right) - \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-R^2)^i}{i!\,(2i+1)} \sum_{j=0}^{i} (C_j^i)^2 \left(\frac{S}{R}\right)^{2j}. \tag{16}$$

From formulas (15), (16) it follows that

$$\Phi_4(R, 0) = \Phi_4(0, R) = \Phi_3(R), \quad \Phi_5(R, 0) = \Phi_5(0, R) = \operatorname{erfc}(R)/R.$$

Let a continuous constant-power heat source q in form of a disk of radius a begins to act in the plane $z = 0$ at the moment $t = 0$. Then at $t > 0$, integrating (13), (14) over the radial coordinate, we arrive at

$$u_z(r, t) = -q\delta A^2 2kt\, \Phi_6(R, A),$$

$$\Phi_6(R, A) \equiv \frac{2}{A^2}\int_0^{A} \Phi_4(R, S)\, SdS, \quad \Phi_7(R, A) \equiv \frac{2}{A^2}\int_0 \Phi_5(R, S)\, SdS. \tag{17}$$

$$T(r, t) = qA^2 \frac{2\sqrt{kt}}{k\rho c}\Phi_7(R, A), \tag{18}$$

The formulas for calculating the functions Φ_6 and Φ_7 are obtained from (17), (18) and (15), (16). For $R \le A$ we have

$$\Phi_6(R, A) = \ln(A/2) + (\gamma + R^2/A^2)/2 +$$

$$+ \sum_{i=1}^{\infty} \frac{(2i-1)!!\,(-A^2)^i}{(2i+2)!!\,i!\,i} \sum_{j=0}^{i} (C_j^i)^2 \frac{(R/A)^{2j}}{(i-j+1)}, \tag{19}$$

$$\Phi_7(R, A) = \frac{2}{\pi A} E(R/A) - \frac{1}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-A^2)^i}{i!\,(2i+1)} \sum_{j=0}^{i} (C_j^i)^2 \frac{(R/A)^{2j}}{(i-j+1)},$$

for $R < A$

$$\Phi_6(R, A) = \ln(R/2) + (1+\gamma)/2 +$$

$$+ \sum_{i=1}^{\infty} \frac{(2i-1)!!\,(-R^2)^i}{(2i+1)!!\,i!\,i} \sum_{j=0}^{i} (C_j^i)^2 \frac{(R/A)^{2j}}{(j+1)},$$

$$\Phi_7(R, A) = \frac{2}{\pi R} K(A/R) + \frac{2}{\pi R}(R^2/A^2)\,[E(A/R) - K(A/R)] -$$

$$- \frac{1}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-R^2)^i}{i!\,(2i+1)} \sum_{j=0}^{1} (C_j^i)^2 \frac{(A/R)^{2j}}{(j+1)}. \tag{20}$$

From relations (19), (20) it follows that

$$\Phi_6(R, 0) = \Phi_3(R), \quad \Phi_7(R, 0) = \mathrm{erfc}(R)/(2R),$$

$$\Phi_6(0, A) = \ln(A/2) + \gamma/2 + \sum_{i=1}^{\infty} \frac{(2i-1)!!\,(-A^2)^i}{(2i+1)!!\,(i+1)!\,i},$$

$$\Phi_7(0, A) = \frac{1}{A} - \frac{1}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-A^2)^i}{i!\,(2i+1)}.$$

It is seen that as $A \to \infty$ the function $\Phi_6(0, A)$ grows without bound. Therefore, at large values of A we determine the asymptotics of the function $\Phi_6(0, A)$. Since

$$\Phi_6(0, A) \equiv \frac{2}{A^2} \int_0^A \Phi_4(0, S)\, S dS = \frac{2}{A^2} \int_0^A \Phi_3(S)\, S dS$$

and

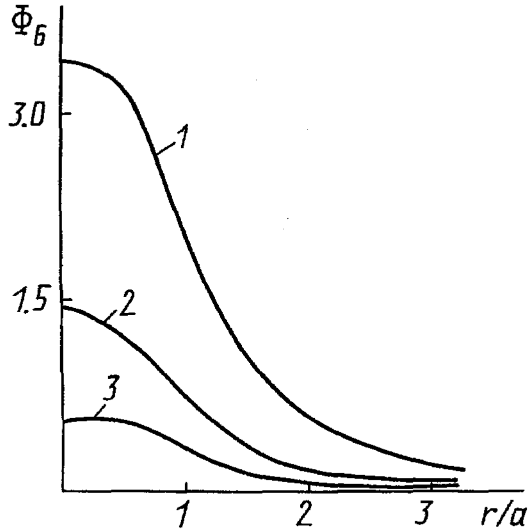
$$\int_0^{\infty} \Phi_3(S)\, S dS = -\pi,$$

then

$$\Phi_6(0, A) = -\frac{2\pi}{A^2} - \frac{2}{A^2} \int_A^{\infty} \Phi_3(S)\, S dS. \tag{21}$$

Using asymptotic expansion (5) of the function $\Phi_3(R)$, we find from (21)

$$\Phi_6(0, A) \cong -\frac{2\pi}{A^2} + \frac{2}{\sqrt{\pi}} \sum_{i=0}^{N} \frac{((2i-1)!!)^2}{(2i+3)\,4^i\,i!\,A^{2i+3}}.$$



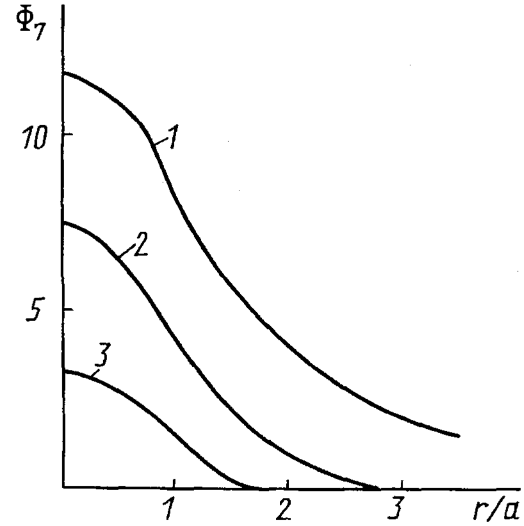


Fig. 1. Φ_6 vs r/a for $a = 0.25$ (1), 0.5 (2), and 1 (3).

Fig. 2. Φ_7 vs r/a for $a = 0.25$ (1), 0.5 (2), and 1 (3).

Figure 1 presents the dependence of $\Phi_6(R, A)$ and Fig. 2 that of $\Phi_7(R, A)$ on the ratio r/a for three values of the radius a of the circular heating region. Calculations were made using formulas (19)–(21).

Application. Employing fundamental solutions (1) and (3) for an instantaneous point source, we may represent the thermal distortion and temperature of the surface $z = 0$ of the semi-space due to the action of heat sources distributed on it with density $q(s, t)$, $s \leq a(t)$, $t > 0$ as

$$u_z(r, t) = \frac{\delta}{4\pi} \int_0^t \int_0^{a(\tau)} \int_0^{2\pi} q(s, \tau)\, \Phi\left(\frac{3}{2}, 2; -\frac{r^2 - 2rs\cos\theta + s^2}{4k(t-\tau)}\right) \frac{s d\theta ds d\tau}{(t-\tau)}, \tag{22}$$

$$T(r, t) = \frac{1}{4\rho c\,(\pi k)^{3/2}} \int_0^t \int_0^{a(\tau)} \int_0^{2\pi} q(s, \tau) \times \frac{\exp\left(-\dfrac{r^2 - 2rs\cos\theta + s^2}{4k(t-\tau)}\right)}{(t-\tau)^{3/2}}\, s d\theta ds d\tau. \tag{23}$$

We now subdivide the interval $[0, t]$ into l parts of length $\delta\tau = t/l$: $0 = \tau_0 < \tau_1 < \ldots < \tau_{l-1} < \tau_l = t$. On $[0, a(\tau)]$ we introduce the uniform grid $0 = a_0 < a_1 < \ldots < a_{n-1} < a_n = a(\tau)$, $a_i = i\delta s$, $i = 0, 1, \ldots, n$; $\delta s = a(\tau)/n$. Let $r_k = a_k - \delta s/2$, $k = 1, 2, \ldots, n$; $t_l = \tau_l - \delta\tau/2$. Assuming the function $q(s, \tau)$ to be constant on each space-time interval $[a_i, a_{i+1}] \times [\tau_j, \tau_{j+1}]$, we find from (19)–(22)

$$u_z(r_k, t_l) \cong \frac{\delta}{4\pi} \sum_{i=1}^{n} \sum_{j=1}^{l} q_{ij}\, c_{ijkl}, \tag{24}$$

$$T(r_k, t_l) \cong \frac{2}{k\rho c} \sum_{i=1}^{n} \sum_{j=1}^{l} q_{ij}\, d_{ijkl} \tag{25}$$

Here

$$c_{ijkl} = -\hat{t}_1 \left[A_{11}^2 \Phi_6(R_1, A_{11}) - A_{12}^2 \Phi_6(R_1, A_{12})\right] +$$

$$+ \hat{t}_2 \left[A_{21}^2 \Phi_6(R_2, A_{21}) - A_{22}^2 \Phi_6(R_2, A_{22})\right], \quad j \neq l,$$

$$c_{ilkl} = \hat{t}_2 \, [A_{21}^2 \, \Phi_6 \, (R_2 \, , \, A_{21}) - A_{22}^2 \, \Phi_6 \, (R_2 \, , \, A_{22}) \,] \, , \quad j = l$$

$$d_{ijkl} = - \sqrt{k \hat{t}_1} \, [A_{11}^2 \, \Phi_7 \, (R_1 \, , \, A_{11}) - A_{12}^2 \, \Phi_7 \, (R_1 \, , \, A_{12}) \,] +$$

$$+ \sqrt{k \hat{t}_2} \, [A_{21}^2 \, \Phi_7 \, (R_2 \, , \, A_{21}) - A_{22}^2 \, \Phi_7 \, (R_2 \, , \, A_{22}) \,] \, , \quad j \neq l \, ,$$

$$d_{ilkl} = \sqrt{k \hat{t}_2} \, [A_{21}^2 \, \Phi_7 \, (R_2 \, , \, A_{21}) - A_{22}^2 \, \Phi_7 \, (R_2 \, , \, A_{22}) \,] \, , \quad j = l$$

$$R_p = \frac{r_k}{2 \sqrt{k \hat{t}_p}} \, ; \quad A_{p1} = \frac{a_i}{2 \sqrt{k \hat{t}_p}} \, , \quad A_{p2} = \frac{a_{i-1}}{2 \sqrt{k \hat{t}_p}} \, , \quad p = 1 \, , \, 2 \, ,$$

$$\hat{t}_1 = (l - j - 1/2) \, \delta \tau \, , \quad \hat{t}_2 = (l - j + 1/2) \, \delta \tau \, , \quad q_{ij} \equiv q \, (r_i \, , \, \tau_j) \, .$$

Relations (24), (25) allow determination of the normal displacements and the temperature of the boundary points of the half-space by using a known law of heat flux distribution $q(s, t)$, $s \leq a(t)$, $t > 0$. If the function $q(s, t)$ is unknown a priori, as is the case in the majority of contact problems of thermoelasticity, then formulas (24), (25) yield a system of linear algebraic equations for determinating the values of $q(s, t)$ at the discrete points r_i, $i = 1, ..., n$, τ_j, $j = 1, ..., l$. The radius of the heating region $a(\tau)$ may be both fixed and changing with time. To determine it, additional physical conditions are employed, namely, the condition of boundedness the contact pressure at the extreme points of the contact region, the absence of interpenetration of the materials of the contacting bodies outside the contact zone, etc.

Finally, it should be noted that the approximating properties of an approximation with the aid of piecewise-constant functions are investigated in [8].

NOTATION

T, temperature; t, time; ρ, density; c, specific heat; k, thermal diffusivity; α, coefficient of linear thermal expansion; E, Young's modulus; ν, Poisson coefficient; erf(*), error function; erfc(*), complementary error function; σ, stress; u, displacement; $I_0(*)$, $I_1(*)$, modified Bessel functions of the first kind; γ, Euler constant; $K(*)$, $E(*)$, total elliptic integrals of the first and second kind, respectively.

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